

Non-gaussianity in axion N-flation models

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We study perturbations in the multi-field axion N-flation model, taking account of the full cosine potential. We find significant differences with previous analyses which made a quadratic approximation to the potential. The tensor-to-scalar ratio and the scalar spectral index move to lower values, which nevertheless provide an acceptable fit to observation. Most significantly, we find that the bispectrum non-gaussianity parameter f_{NL} may be large, typically of order 10 for moderate values of the axion decay constant, increasing to of order 100 for decay constants slightly smaller than the Planck scale. Such a non-gaussian fraction is detectable. We argue that this property is generic in multi-field models of hilltop inflation.

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Much focus has been placed lately on the discovery potential of cosmic non-gaussianity in the statistics of primordial perturbations. The Wilkinson Microwave Anisotropy Probe (WMAP) has already set interesting limits [1]. The Planck satellite, now taking data, will improve these significantly, reaching a sensitivity to the non-gaussianity parameter f_{NL} of around five. Discovery of non-gaussianity would open a new arena of cosmological observations particularly suited to probing early Universe physics.

Present ideas in fundamental physics suggest there may be many scalar fields which can influence the early Universe, including inflation. N-flation [2] uses many string axions to provide a realization of the ‘assisted inflation’ phenomenon [3], in which a collection of scalar fields cooperatively support inflation even if their potentials are individually too steep. The phenomenology of such models (see also Ref. [4]) links fundamental physics and upcoming cosmological observations.

Previous N-flation studies have assumed that all relevant fields are close to their minima and can be described by quadratic potentials. For axions the full potential is trigonometric and we find the quadratic approximation is unreliable. Even for identical potentials, the condition for stable co-evolution of the fields is violated near the hilltop [5]. Therefore fields in this region evolve on divergent trajectories. Accounting for this divergence by retaining the full potential leads to two very significant changes. The predicted scalar spectral index and tensor-to-scalar ratio, r , are reduced. This remains compatible with existing observations but may leave r undetectable. More importantly, f_{NL} is predicted to be large, and very plausibly within the range of future probes.

This unexpectedly large non-gaussianity is a genuine multi-field phenomenon. It is a consequence of the diverging trajectories near the hilltop, implied by a negative η -parameter of order unity or larger. In single-field models, potentials of this form lead to a density perturbation with a spectral index, n , in conflict with obser-

vation. The assisted inflation mechanism reduces $1 - n$ to an acceptable value, but leaves f_{NL} dominated by the contribution of the field closest to the peak.

THE MODEL

The axion N-flation model is based on a set of N_f uncoupled fields, labelled ϕ_i , each with a potential [2]

$$V_i = \Lambda_i^4 (1 - \cos \alpha_i) , \quad (1)$$

where $\alpha_i = 2\pi\phi_i/f_i$ and f_i is the i^{th} axion decay constant. In a more general model couplings may exist between the fields, but we will not consider these. The mass of each field in vacuum satisfies $m_i = 2\pi\Lambda_i^2/f_i$, and the angular field variables α_i lie in the range $(-\pi, +\pi]$. Without loss of generality we will set initial conditions with all α_i positive. If only a single field is present this model is known as natural inflation [6].

Calculation of the observables n , r and f_{NL} makes use of the δN formula [7], which considers how the total number of e -folds of expansion N is modified by field perturbations. We define slow-roll parameters for each field as

$$\epsilon_i \equiv \frac{M_{\text{P}}^2}{2} \left(\frac{V'_i}{V_i} \right)^2 , \quad (2)$$

where $M_{\text{P}} \equiv (8\pi G)^{-1/2}$ is the reduced Planck mass, a prime denotes the derivative of a function with respect to its argument, and no summation over i is implied. The global slow-roll parameter $\epsilon \equiv -\dot{H}/H^2$ can be written as a weighted sum $\epsilon \simeq \sum_i (V_i/V)^2 \epsilon_i$, in which each field contributes according to its share of the total energy density. We must have $\epsilon < 1$ during inflation.

We work in the horizon-crossing approximation, in which the dominant contribution to each observable is assumed to arise from fluctuations present only a few e -folds after horizon exit of the wavenumber under discussion. After smoothing the universe on a superhorizon scale somewhat smaller than any scale of interest,

the horizon-crossing approximation becomes valid whenever the ensemble of trajectories followed by smoothed patches of the universe approaches an attractor. We suppose that inflation exits gracefully, with each field settling into the minimum of its potential. The horizon-crossing formulas will then be a reasonable approximation. Using this method, and conventional definitions for each observable parameter [8], we find

$$\mathcal{P}_\zeta = \frac{H_*^2}{4\pi^2} \sum_i N_{,i} N_{,i} = \frac{H_*^2}{8\pi^2 M_{\text{P}}^2} \sum_i \frac{1}{\epsilon_i^*}; \quad (3)$$

$$n-1 = -2\epsilon_* - \frac{8\pi^2}{3H_*^2} \sum_j \frac{\Lambda_j^4}{f_j^2 \epsilon_j^*} / \sum_i \frac{1}{\epsilon_i^*}; \quad (4)$$

$$r = \frac{2}{\pi^2 \mathcal{P}_\zeta} \frac{H_*^2}{M_{\text{P}}^2} = 16 / \sum_i \frac{1}{\epsilon_i^*}; \quad (5)$$

$$\frac{6}{5} f_{\text{NL}} \simeq \frac{\sum_{ij} N_{,i} N_{,j} N_{,ij}}{(\sum_k N_{,k} N_{,k})^2} = \frac{r^2}{128} \sum_i \frac{1}{\epsilon_i^*} \frac{1}{1 + \cos \alpha_i^*}, \quad (6)$$

where $N_{,i}$ and $N_{,ij}$ are respectively the first and second derivatives of N with respect to the fields, and $*$ indicates evaluation at horizon crossing (determined by Eq. (7) below). In writing Eq. (6) any intrinsic non-gaussianity among the field perturbations at horizon crossing has been neglected, a good approximation provided $f_{\text{NL}} > 1$ [9, 10]. Our sign convention for f_{NL} matches WMAP [1], and the non-gaussianity is predicted to be of local type. The observed amplitude of perturbations is obtained by adjusting the Λ_i to give an appropriate value of H_* .

Under a quadratic approximation to each potential, it can be shown that Eqs. (5) and (6) recover their single-field values of order $\sim 1/N_*$ [10, 11], making f_{NL} undetectably small. The spectral index can be shown to be less than its single field value $1 - 2/N_*$ [12] with equality only in the equal-mass case. Its value for a given choice of parameters must be computed numerically [13]. However, we will see that these results all change whenever our initial conditions populate the hilltop region.

N-FLATION PERTURBATIONS

Eqs. (3)–(6) apply for any choice of Λ_i and f_i . We restrict attention to the case where all fields have the same potential, which already captures the interesting phenomenology. A broader investigation will be published elsewhere. The scale $\Lambda \equiv \Lambda_i$ is fixed from the observed amplitude of \mathcal{P}_ζ , leaving $f \equiv f_i$ and N_{f} as adjustable parameters. The initial conditions are drawn randomly from a uniform distribution of angles α_i , with several realizations to explore the probabilistic spread. From these two parameters we predict the observables n , r and f_{NL} .

There are two constraints. First, we require sufficient e -foldings. For a given set of initial angles α_i , and ignoring a small correction from the location of the end of

inflation, one finds

$$N_{\text{tot}} \simeq \sum_i \left(\frac{f_i}{2\pi M_{\text{P}}} \right)^2 \ln \frac{2}{1 + \cos \alpha_i} \simeq \frac{\ln 2}{2\pi^2} \frac{f^2}{M_{\text{P}}^2} N_{\text{f}}, \quad (7)$$

where in the second equality we have replaced N_{tot} with its expectation value by averaging over α_i . Eq. (7) is replicated to high accuracy in numerical simulations. For a given f it determines the minimum number of fields required for sufficient inflation, typically several hundred or more. There is no similar constraint from the spectral index. When $N_{\text{tot}} \approx N_*$, the α_i^* are uniformly distributed and $\langle n-1 \rangle \simeq -5 \ln 2 / N_*$, independent of f and N_{f} . This tilt is observationally acceptable. For larger N_{f} the spectral index approximately satisfies Eq. (8) below.

Second, a key motivation of the N-flation model was to obviate the requirement for superplanckian field values, which are invoked in many single-field models. If one literally imposes $|\phi| < M_{\text{P}}$ this requires $f_i < 2M_{\text{P}}$ for each i . However, it would be reasonable to regard this condition as approximate and not mandatory.

The ϵ_i approach zero for fields close to the hilltop, so each summation in Eqs. (3)–(6) is dominated by those fields with the smallest ϵ_i . Suppose some number \bar{N} of fields have roughly comparable ϵ_i , of order $\bar{\epsilon}$. The observable parameters have different scalings with \bar{N} . The spectrum, \mathcal{P}_ζ , scales like \bar{N} copies of a single-field model with slow-roll parameter $\bar{\epsilon}$, whereas r is reduced by a factor \bar{N} compared to its value in the same single-field model. The spectral index can be written exactly (within slow-roll) in terms of a single sum coming from H_* ,

$$n-1 \approx -2\epsilon_* - 8\pi^2 \left(\frac{M_{\text{P}}}{f} \right)^2 / \sum_i (1 - \cos \alpha_i^*), \quad (8)$$

and is independent of \bar{N} . It becomes close to $-2\epsilon_*$ when the denominator is of order 10^3 . This is the standard assisted-inflation mechanism. Most importantly, f_{NL} has the approximate behaviour

$$\frac{6}{5} f_{\text{NL}} \approx \frac{2\pi^2}{\bar{N}} \left(\frac{M_{\text{P}}}{f} \right)^2, \quad (9)$$

which is independent of $\bar{\epsilon}$ if the dominant fields are sufficiently close to the hilltop. N-flation has lifted the single-field consistency condition $f_{\text{NL}} \approx -(5/12)(n-1)$ [9, 10], which prevents single-field models generating large non-gaussianity without violating observational bounds on n .

Where the summations in Eqs. (3)–(6) are dominated by a single field, this formula shows that f_{NL} can become rather large, scaling as $(M_{\text{P}}/f)^2$. For $f = M_{\text{P}}$, we find $f_{\text{NL}} \lesssim 16.4$; a non-gaussian fraction of this magnitude should be visible to the Planck satellite. It is even possible to achieve $f_{\text{NL}} \sim 100$ for $f \sim 0.4M_{\text{P}}$, though then N_{f} must be very large to gain sufficient e -foldings. If $f_{\text{NL}} \gtrsim 50$ it may be more profitable for Planck to search for non-linearity in the trispectrum [14], for which estimates in

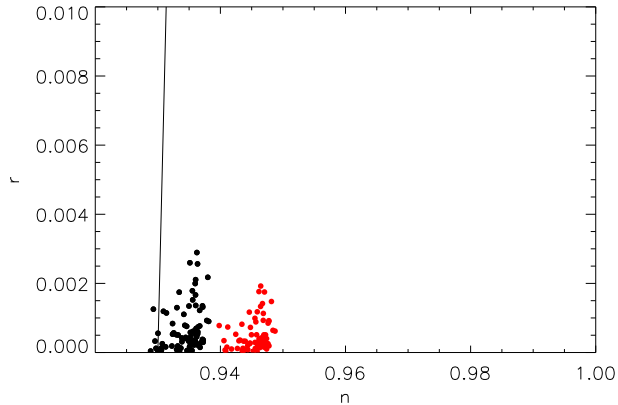


FIG. 1: Predictions in the n - r plane, averaged over realizations, for various values of f between $0.4M_P$ and $2M_P$ and of N_f between 464 and 10,000, all giving sufficient inflation. The black (left) cluster of points takes $N_* = 50$ and the red (right) cluster $N_* = 60$. The quadratic expansion predicts $r = 8/N_*$, far off the top of this plot. The region right of the line is within the WMAP7+BAO+ H_0 95% confidence contour [1].

the quadratic approximation were given in Ref. [15]. We defer a full analysis of the trispectrum to future work but note that the trispectrum equivalents of Eq. (9) are, in conventional notation [16], $\tau_{\text{NL}} = (4\pi^4/\bar{N}^2)(M_P^4/f^4)$ and $(54/25)g_{\text{NL}} = (8\pi^4/\bar{N}^2)(M_P^4/f^4)$.

The expectations described above are borne out in numerical calculations. In Fig. 1 we show model predictions in the n - r plane, averaged over several realizations of the initial conditions. We see n and r are only weakly dependent on the model parameters (though there is significant dispersion amongst realizations, not shown here), with the choice of N_* being the principal determinant of n . In Fig. 2 we plot f_{NL} as a function of N_f for $f = M_P$, with ten realizations at each N_f . This clearly shows the expected maximum, which is nearly saturated in cases where a single field dominates the summations. In cases where several fields contribute significantly to the sums in Eqs. (3)–(6), the non-gaussian fraction is reduced. Fig. 3 shows the mean predicted non-gaussianity, averaged over realizations, as a function of f .

Eqs. (8) and (9) clarify the origin of large f_{NL} in this model. The cooperative effect of the N-flation mechanism does not enhance the non-gaussian signal. Indeed, f_{NL} is suppressed by the central limit theorem where $\bar{N} \gg 1$ fluctuations contribute equally to the curvature perturbation. Nor does the large effect arise from a singularity in the e -folding history, N , as a function of its initial angles α_i . Although Eq. (7) is singular in the limit $\alpha_i \rightarrow \pi$, its Taylor expansion is trustworthy unless $|\alpha_i - \pi| \lesssim (\mathcal{P}_\zeta r)^{1/2}(M_P/f_i)$. The observed magnitude of \mathcal{P}_ζ requires $|\alpha_i - \pi| \gtrsim r^{1/2}(f_i/M_P)$ for each field, so a breakdown of the Taylor expansion cannot become

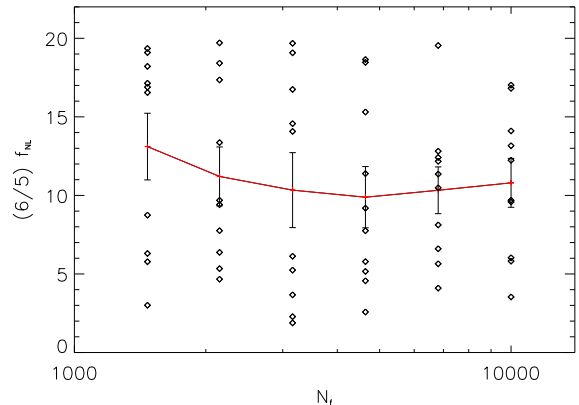


FIG. 2: Predicted non-gaussianity, $\frac{6}{5}f_{\text{NL}}$, for $f = M_P$ and $N_* = 50$. The error bars are on the mean over realizations (not the standard deviation). Here the maximum achievable value of $\frac{6}{5}f_{\text{NL}}$ is $2\pi^2 \simeq 20$, almost saturated in some realizations. The significant spread is due to initial condition randomness with typical mean values being around half the maximum achievable value, and no discernable trend with N_f .

relevant unless at least one f_i is a few orders of magnitude less than the Planck scale, of order $(f_i/M_P)^4 \lesssim \mathcal{P}_\zeta$. These constraints additionally imply that we do not trespass on any region of field space where quantum diffusion competes with classical motion.

Instead, the large f_{NL} derives from a generic dispersive effect present in any hilltop potential. Measuring the displacement of ϕ_i from the hilltop by δ_i , each potential can be approximated in its vicinity by $V_i \approx 2\Lambda_i^4(1 + \eta_i\delta_i^2/2M_P^2)$, where $\eta_i < 0$ satisfies

$$\eta_i \equiv M_P^2 \frac{V_i''}{V_i} \simeq -2\pi^2 \left(\frac{M_P}{f_i} \right)^2. \quad (10)$$

These potentials are tachyonic. Fields close to the hilltop remain almost stationary, while fields further away are ejected downhill. This process typically leaves a few fields on top of the hill, which have small ϵ_i and dominate the sums in Eqs. (3)–(6). It seems clear this behaviour is generic for any N-flation model constructed using hilltop potentials. The few fields remaining in the vicinity of the hilltop each generate contributions to the curvature perturbation with third moment $(6/5)f_{\text{NL}} \approx -\eta_*$ [9]. Accounting for suppression arising from the central limit theorem, we recover the approximate expression (9). For a general hilltop potential, well-rehearsed arguments lead us to expect $|\eta| \sim 1$ and therefore $f_{\text{NL}} \sim 1$. In a single-field model this is the ‘ η problem’. In an N-flation model, it is a generic expectation of enhanced non-gaussianity. Even larger yields are possible in some models, including our case, if it is possible to achieve $|\eta| \gg 1$ while preserving technical naturalness.

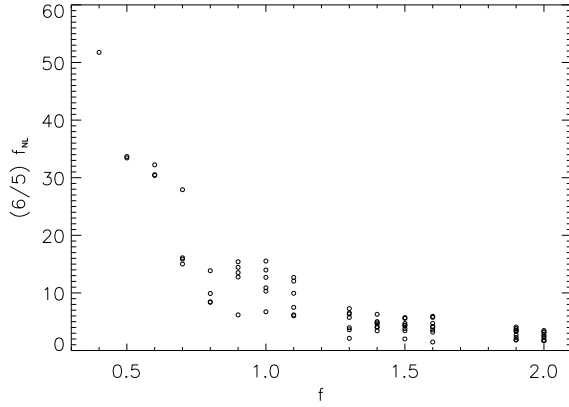


FIG. 3: The predicted non-gaussianity as a function of f , for a range of choices of N_f . Each point shown is the average of five or more realizations for an f - N_f pair. We see a strong trend with f , well represented by Eq. (9) with $\bar{N} \simeq 2$. The different N_f are scattered by randomness in the initial conditions rather than an identifiable trend.

CONCLUSIONS

We have described a new mechanism for generating observably-large cosmic non-gaussianity, based on the strongly-dispersive dynamics of fields in a hilltop region. In a multi-field context such as the axion N-flation model, assisted inflation can yield a viable spectral index without a major dilution of the non-gaussianity. As compared to the quadratic potential approximation to N-flation, we found a substantial decrease in r , a modest increase in $1 - n$, and a substantial increase in f_{NL} . These changes will happen whenever initial conditions have a significant probability of populating the hilltop region, such as the uniform (in field angle) initial conditions we chose.

Searches have previously been made for models which achieve $|f_{NL}| \gg 1$ while preserving slow-roll during inflation [17]. The N-flation model is of this type, but offers several advantages. The non-gaussian fraction is naturally bounded above, so that f_{NL} cannot become arbitrarily large. Therefore our predictions do not depend on a sudden exit from inflation, e.g. triggered by a hybrid transition, to prevent f_{NL} from growing to an unacceptable value. Equally important, our large signal does not derive from a singularity of the e -folding history N , as a function of its initial conditions. These means we can rely on a perturbative expansion. We can simultaneously satisfy observational constraints on the spectral index and tensor fraction. Moreover, this result seems generic. Inflation is self-replicating on top of the hill, sometimes described as ‘topological inflation’ [18]. Coupled with the dispersion of trajectories originating from the vicinity of the hilltop, this implies large non-gaussianity may not be uncommon over a landscape of scalar field vacua.

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